UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3506**

MODULE NAME : **Mathematics in Biology I**

DATE : **27-Apr-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH3506A-001-EXAM-25

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the terms *generation* and *life expectancy* for a species (you are not asked for mathematical expressions for them). What does it mean to say a population is viable?

   (b) Explain what is meant by non-overlapping generations, and why a population with non-overlapping generations can be modelled using a recurrence relation.

   (c) Consider the following model giving the population density, $N_{k+1} \geqslant 0$, of a population at time $k+1$ in terms of the population density at time k:

$$N_{k+1} = N_k \exp\left\{r\left(1 - \frac{N_k}{K}\right)\right\}, \quad r, K > 0, \quad k = 0, 1, \ldots. \qquad (1)$$

   Find all the steady states of (1) and determine their stability for each value of $r > 0$.

   (d) Sketch a cobweb map (i.e. iterates of N_k versus N_{k+1}) for the cases (i) $0 < r < 1$, (ii) $1 < r < 2$ and (iii) $r > 2$.

2. For δt small let $b(t)\delta t + O(\delta t^2)$ be the probability that an individual gives birth in the time interval $[t, t+\delta t)$. Show that the probability $P(t)$ that no individual gives birth in $[0, t)$ is given by

$$P(t) = \exp\left(-\int_0^t b(s)\,\mathrm{d}s\right)$$

Now suppose that the birth rate is $b(t) = te^{-at}$ for $a > 0$.

   (a) What is the probability that an individual will give birth sometime in their lifetime?

   (b) What is the probability given that an individual does gives birth in the time interval $[0, t)$ given that it gives birth sometime?

[You may ignore deaths].

3. (a) Consider the logistic equation for population growth:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right).$$

(i) What do r and K represent ecologically?

(ii) By considering the function $V(N) = K\log(K/N) + N - K$, show that for all $N(0) \neq 0$ solutions $N(t) \to K$ as $t \to \infty$.

(b) The interactive behaviour between two species is modelled by the coupled system

$$\begin{aligned} \frac{\mathrm{d}N_1}{\mathrm{d}t} &= r_1 N_1\left(1 - \frac{N_1}{K_1 + b_{12}N_2}\right) \\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= r_2 N_2\left(1 - \frac{N_2}{K_2 + b_{21}N_1}\right) \end{aligned}$$

where $r_1, r_2, b_{12}, b_{21}, K_1, K_2$ are all positive parameters.

(i) Find all steady states and determine their local stability.

(ii) Sketch the (N_1, N_2)−phase plane when $b_{12}b_{21} < 1$ and interpret your results ecologically.

4. A disease agent invades a large population. Initially all members of the population are susceptible to the disease, and the disease is fatal. The per capita birth rate for the uninfected population is $b > 0$ (a constant); an infected individual cannot give birth. In the absence of the disease the normal per capita death rate is $d > 0$ (a constant). The per capita death rate due to the infection is $\delta > 0$.

(a) Denoting the susceptible population density by S and the infective population density by I, and assuming that the rate of infection of susceptibles is λSI with $\lambda > 0$, obtain equations for the population dynamics of I and S.

(b) Hence show that if $N = I + S$,

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= -(b+\delta)I + (b-d)N \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= -(d+\delta)I + \lambda(N-I)I. \end{aligned} \qquad (2)$$

(c) Sketch the (N, I)−phase plane when $b < d$.

(d) Find a function $H(N, I)$ such that H is constant along a solution of (2) and hence, or otherwise, sketch the (N, I)−phase plane in the case $b > d$. [Hint: first find a function $h(S, I)$ that is constant along solutions $S(t), I(t)$.]

5. In an age-structured population there are n age classes, the population density of age k at time t is denoted by $N_k(t)$ and $\mathbf{N}(t) = (N_1(t), ..., N_n(t))^T$. The expected number of offspring of a female at age k is b_k and the probability that an individual aged $k \geqslant 0$ (with $k = 0$ the newborns) survives to age $k+1$ is p_k. No individual can survive past age n.

   (i) Show that $\mathbf{N}(t+1) = L\mathbf{N}(t)$ for $t = 1, 2, ...$ where L is a real $n \times n$ matrix which you should find.

   (ii) Show that the eigenvalues λ of L satisfy

   $$\sum_{r=1}^{n} \frac{b_r \ell_r}{\lambda^r} = 1,$$

   where $\ell_k = \Pi_{i=0}^{k-1} p_k$ for $0 \leqslant k \leqslant n$.

   (iii) Show that L has a unique real positive eigenvalue λ.

   In an age-structured population, juveniles J reach sexual maturity at age $m > 1$ (and thus become adults A). Before reaching adulthood they survive from one year to the next with probability p_J. Adults survive from one year to the next with probability p_A, and their expected number of offspring per year is $b > 0$, but adults cannot live older than age n. Show that the mean fitness λ satisfies

   $$\lambda^{n+1} - p_A \lambda^n - b p_J^m \lambda^{n-m+1} + b p_J^m p_A^{n-m+1} = 0.$$

END OF PAPER